EVERY REAL BANACH SPACE CAN BE RENORMED TO SATISFY THE DENSENESS OF NUMERICAL RADIUS ATTAINING OPERATORS

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ABSTRACT

First we show that every real Banach space satisfying a certain property, called β (used by Lindenstrauss and Partington) verifies the denseness of the numerical radius attaining operators. Using this result and a renorming theorem by Partington we conclude that every Banach space is isomorphic to a new one satisfying the denseness of the numerical radius attaining operators.

We deal with a "Bishop–Phelps" type problem, raised by B. Sims in his doctoral dissertation [16]. To introduce this question, let us recall the following notions: Let X be a Banach space and $T \in L(X)$ (where $L(X)$ will denote the space of all bounded and linear operators on X). The **numerical range** of T, $V(T)$, is the set of scalars given by

$$V(T) = \{f(T(x)) : (x, f) \in \Pi(X)\}$$

where

$$\Pi(X) = \{(x, f) \in S_X \times S_{X^*} : f(x) = 1\}.$$

(S_X and S_{X^*} are the unit spheres of X and its dual, respectively.)
The **numerical radius** of T, $v(T)$, is defined by

$$v(T) = \mathrm{Sup}\{|\lambda| : \lambda \in V(T)\}.$$

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We say that T **attains its numerical radius** if there exists $\lambda \in V(T)$ such that $v(T) = |\lambda|$, that is, when the above supremum is actually a maximun. We will denote by $R(X)$ the set of all (bounded and linear) operators on X which attain their numerical radii. A complete survey on the concept and properties of numerical ranges can be found in the monographs by Bonsall-Duncan [5,6].

The kind of problem Sims raised can be posed in the following way:

Is it true, for any Banach space X, that the numerical radius attaining operators on X is a dense set in $L(X)$?

In 1972 B. Sims gave the first answer [16] showing that, on a Hilbert space, the set of selfadjoint operators which attain their numerical radii is dense in the set of all self-adjoint operators. Then, in 1984 I. Berg and B. Sims [4] proved that uniform convexity on the Banach space is enough to get a positive answer. Using this result, two years later, C. Cardassi proved the denseness of the set $R(X)$ when X is a uniformly smooth space [9] and solved the problem for the spaces c_0, ℓ_1 [8], $C(K)$ [7] (real functions) and $L_1(\mu)$ (where μ is a regular and positive Borel measure on a compact and Hausdorff topological space K) [10].

Then, in 1989, R. Payá and the author, proved, for any Banach space X, the denseness of the set of operators on X whose adjoints attain their numerical radii. As a consequence, if X is reflexive, they gave a positive answer to the problem by Sims [1,2]. The same authors also obtained a condition on the Banach space X to guarantee the denseness of $R(X)$, more general than reflexivity, which is Radon–Nikodym property.

In 1990, the author proved that every weakly compactly generated space is isomorphic to a Banach space Y for which $R(Y)$ is dense in $L(Y)$ and also obtained that every Banach space X is linearly isometric to a 1-complemented subspace of a Banach space Y with the same density character and such that the numerical radius attaining operators are dense [3]. Anyway, as the property of denseness of $R(X)$ did not seem to be hereditary, the original problem still remained opened. In 1991, R. Payá [13] gave an example of a Banach space X for which $R(X)$ is not dense in $L(X)$, solving the question by Sims. Using this counterexample and the above embedding result, we can deduce that the denseness of $R(X)$ in $L(X)$ is highly nonstable under subspaces.

Since there is a counterexample, we could think on the isomorphic version of this problem. In this paper, we prove that, every Banach space X is isomorphic to a Banach space Y which verifies $\overline{R(Y)} = L(Y)$. To get this result, we show, in fact, that a certain geometrical condition, called property β, implies denseness of numerical radius attaining operators. The isometric result is also interesting by itself and proves, once more, the paralelism between denseness of norm attaining operators and numerical radius attaining operators, even when these two sets are not necessarily the same.

In the following every Banach space considered will be real.

First we recall property β.

Definition 1 [11]: A Banach space has **property** β if there exists a subset $\{(x_\alpha, f_\alpha) : \alpha \in \Lambda\}$ of $S_X \times S_{X^*}$ satisfying the following assertions

(i) $f_\alpha(x_\alpha) = 1$.

(ii) There exists λ, $0 \le \lambda < 1$ and such that $|f_\alpha(x_\beta)| \le \lambda$ for $\alpha \ne \beta$.

(iii) For every x in X, $\|x\| = \text{Sup}\{|f_\alpha(x)| : \alpha \in \Lambda\}$.

Note that this definition generalizes the geometric behaviour of the elements $\{(e_n, e_n^*) : n \in \mathbb{N}\}$ in c_0 and ℓ_∞. Property β was already used by J. Lindenstrauss [11] and W. Schachermayer [15] related with norm attaining operators. In fact, Lindenstrauss proved that if X has property β, for every Banach space Y, the norm attaining operators from Y to X is a dense set in $L(Y, X)$ (bounded and linear operators from Y to X) [11].

We will prove the analogous result of one by Lindenstrauss for numerical radius attaining operators. The proof of this fact follows the original paper by Lindenstrauss, but the adaptation to numerical radius needs non trivial arguments to apply his technique. First of all, we will try to compute the numerical radius of an operator $T \in L(X)$, being X a Banach space with property β. To get this kind of relation, we will prove that the unit sphere can be generated by the faces determined by the elements $\{f_\alpha\}$ which appear in the definition of property β. Let us denote by $\mathbb{T} = \{1, -1\}$ and by B_X the closed unit ball of X.

LEMMA 2: *Let X be a Banach space with property β, then the set $\bigcup_{\alpha\in\Lambda} \mathbb{T}F_\alpha$ is a dense set of S_X, where $F_\alpha = \{x \in B_X : f_\alpha(x) = 1\}$.*

Proof: Fix $x \in S_X$ and $\varepsilon > 0$. We choose $0 < \gamma < \frac{\varepsilon}{4}$ and consider the real function $\varphi : [0,1] \longrightarrow \mathbb{R}$ defined by

$$\varphi(t) = (1-\gamma)(1+t\lambda) + \gamma\lambda \qquad (0 \le t \le 1),$$

where λ is the real number which satisfies (ii) in the definition of property β. φ

satisfies $\varphi(0) = 1 + \gamma(\lambda - 1) < 1$, so there exists $0 < \delta < \min\{1, \frac{\varepsilon}{4}\}$ such that for $0 \le t < \delta$ we have $\varphi(t) < 1$.

Now, by condition (iii) in property β, we can choose $\alpha \in \Lambda$ such that $|f_\alpha(x)| > 1 - \delta$, and let $w = \text{sign } f_\alpha(x)$; taking $t_0 = 1 - f_\alpha(wx)$ we get $\varphi(t_0) < 1$.

Define now $y = wx + t_0 x_\alpha$, so $\|y\| \le 1 + t_0$ and $f_\alpha(y) = 1$ by the choice of t_0 and the fact that $f_\alpha(x_\alpha) = 1$; taking $z = (1-\gamma)y + \gamma x_\alpha$, one has, of course, $f_\alpha(z) = 1$ and we want to prove that $z \in F_\alpha$. To get this, we just need z to be an element in B_X.

Using again that the set $\{f_\beta\}$ is norming (condition (iii)), it is enough to compute $f_\beta(z)$. For $\beta \neq \alpha$, we have

$$|f_\beta(z)| = |(1-\gamma)f_\beta(y) + \gamma f_\beta(x_\alpha)|$$

$$= |(1-\gamma)\left[w f_\beta(x) + t_0 f_\beta(x_\alpha)\right] + \gamma f_\beta(x_\alpha)| \quad \text{(by Definition 1(ii))}$$

$$\le (1-\gamma)(1 + t_0\lambda) + \gamma\lambda = \varphi(t_0) < 1.$$

As we had $f_\alpha(z) = 1$, then $z \in F_\alpha$. Now, we check that z is near to wx; just using the definition of z, y and the choice of γ and t_0, we have

$$\|z - wx\| = \|(1-\gamma)y + \gamma x_\alpha - wx\| = \|(1-\gamma)y + \gamma x_\alpha - y + t_0 x_\alpha\|$$

$$\le \gamma\|x_\alpha - y\| + t_0\|x_\alpha\| \le \gamma(\|x_\alpha\| + \|y\|) + t_0\|x_\alpha\|$$

$$\le \gamma(2 + t_0) + t_0 \le \gamma(2+\delta) + \delta \le 3\gamma + \delta \le 3\frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Note that we showed that $wz \in \mathbb{T}F_\alpha$ and $\|wz - x\| < \varepsilon$ and this concludes the proof. ∎

From the above result it is easy to get an estimate for the numerical radius of operators on Banach spaces with property β.

LEMMA 3: *Let X be a Banach space with property β and $T \in L(X)$, then*

$$v(T) = \text{Sup}\{|f_\alpha(T(x))| : \alpha \in \Lambda, x \in F_\alpha\}.$$

Proof: By the above lemma the set $\cup_{\alpha\in\Lambda}\mathbb{T}F_\alpha$ is a dense set in the unit sphere of X, so we can apply [5; Theorems 9.3 and 9.4], which gives

$$v(T) = \text{Sup}\{|w f_\alpha(T(wx))| : w \in \mathbb{T}, \alpha \in \Lambda, x \in F_\alpha\},$$

but the above supremum is clearly the same as

$$\text{Sup}\{|f_\alpha(T(x))| : \alpha \in \Lambda, x \in F_\alpha\}. \quad ∎$$

To prove the announced result we also need the Bishop–Phelps Theorem:

THEOREM 4 ([14; Proposition 7]): *Let X be a Banach space and C be a non-empty, closed convex and bounded set of X, then the set*

$$\{f \in X^* : |f| \text{ attains its supremum on } C\}$$

is dense in X^*.

We can now prove our main theorem.

THEOREM 5: *Let X be a Banach space with property β. Given $T \in L(X)$, $\varepsilon > 0$, there exists an operator $S \in L(X)$ with $\|S - T\| < \varepsilon$, such that $S - T$ is a rank-one operator and $S \in R(X)$. In particular, $R(X)$ is a dense set in $L(X)$.*

Proof: We will assume $\|T\| < 1$ and choose $\eta > 0$ such that

$$\eta < \min\left\{\frac{\varepsilon}{3}, \frac{1-\lambda}{2}, 1 - \|T\|\right\}. \tag{1}$$

First we will perturb the initial operator T to get a new operator T_1, such that the numerical radius of T_1 can be estimated using just a fixed index $\alpha \in \Lambda$ in the supremun appearing in Lemma 3.

By Lemma 3, we can choose $\alpha \in \Lambda$ and $x \in F_\alpha$ such that

$$|f_\alpha(T(x))| > v(T) - \eta^2. \tag{2}$$

Define

$$T_1(z) = T(z) + \eta w f_\alpha(z) x_\alpha \qquad (z \in X),$$

where $w = \operatorname{sign} f_\alpha T x$.

Clearly $T_1 \in L(X)$, $\|T_1 - T\| \leq \eta$. We claim that

$$v(T_1) = \operatorname{Sup}\{|f_\alpha(T(z))| : z \in F_\alpha\}$$

where α is the index satisfying (2). If we choose $\beta \in \Lambda$, $\beta \neq \alpha$ and $z \in F_\beta$, we will have

$$|f_\beta(T_1(z))| = |f_\beta(T(z)) + \eta w f_\alpha(z) f_\beta(x_\alpha)| \leq v(T) + \eta\lambda, \tag{3}$$

where we used $|f_\beta(x_\alpha)| \leq \lambda$ for $\alpha \neq \beta$. However, using (2) we get

$$|f_\alpha(T_1(x))| = |f_\alpha(T(x)) + \eta w| = |f_\alpha(T(x))| + \eta \geq v(T) - \eta^2 + \eta. \tag{4}$$

By the choice of $\eta, (\eta < \frac{1-\lambda}{2})$ $v(T) + \eta\lambda + \eta^2(1-\lambda) \leq v(T) - \eta^2 + \eta$, and this implies, in view of (3) and (4)

$$\sup_{\beta \neq \alpha}\{|f_\beta(T_1(z))| : z \in F_\beta\} + \eta^2(1-\lambda) \leq \sup\{|f_\alpha(T_1(z))| : z \in F_\alpha\}$$

so the number in the right side is $v(T_1)$, by Lemma 3.

Up to this point we got an operator T_1 such that

$$\|T - T_1\| \leq \eta, \qquad \|T_1\| \leq \|T\| + \eta < 1$$

and there exists $\alpha \in \Lambda$ satisfying

$$v(T_1) = \sup_{z \in F_\alpha} |f_\alpha(T(z))| \geq \sup_{\beta \neq \alpha} \{|f_\beta(T_1(z))| : z \in F_\beta\} + \eta^2(1-\lambda). \tag{5}$$

Now we will construct the desired operator S. Choose $0 < \gamma \leq \eta^2$ and we apply Theorem 4 taking F_α as C and find g in the unit ball of X^*, such that

$$\|g - T_1^*(f_\alpha)\| < \gamma^3 \tag{6}$$

and $|g|$ attains its supremum on F_α. Define

$$S(z) = T_1(z) + \left[(1+\gamma^2)g(z) - T_1^*(f_\alpha)(z)\right] x_\alpha \qquad (z \in X). \tag{7}$$

Of course, S is a linear and bounded operator on X.

First we prove that S attains its numerical radius. To show this, let us note that for $\beta \in \Lambda, \beta \neq \alpha$, $z \in F_\beta$, using (5) we get

$$|f_\beta(S(z))| \leq v(T_1) - \eta^2(1-\lambda) + \lambda(\gamma^3+\gamma^2) \leq v(T_1) - \gamma(1-\lambda) + \lambda(\gamma^3+\gamma^2), \tag{8}$$

while for $z \in F_\alpha$, $f_\alpha(S(z)) = (1+\gamma^2)g(z)$ and so, using (6) and (5)

$$\sup_{z \in F_\alpha} |f_\alpha(S(z))| \geq (1+\gamma^2) \sup_{z \in F_\alpha} |g(z)|$$

$$\geq (1+\gamma^2)\left[\sup_{z \in F_\alpha} |T_1^*(f_\alpha(z))| - \gamma^3\right] \geq (1+\gamma^2)\left[v(T_1) - \gamma^3\right]. \tag{9}$$

By the choice of γ, $\gamma \leq \frac{1-\lambda}{4}$ and so

$$v(T_1) \geq 0 > \gamma^3 + \gamma + \lambda(1+\gamma) - \frac{1}{\gamma}(1-\lambda),$$

which is equivalent to

$$(1+\gamma^2)\left(v(T_1)-\gamma^3\right) > v(T_1) - \gamma(1-\lambda) + \lambda(\gamma^3+\gamma^2).$$

In view of (8) and (9) the last inequality implies that

$$\sup\{|f_\beta(S(z))| : z \in F_\beta, \beta \neq \alpha\} \leq \sup\{|f_\alpha(S(z))| : z \in F_\alpha\},$$

so using again Lemma 3 the supremum on the right is $v(S)$, which is actually a maximum, because

$$f_\alpha(S(z)) = (1+\gamma^2)g(z) \qquad (z \in F_\alpha)$$

and $|g|$ attains its supremum on F_α, that is, S attains its numerical radius.

We also have

$$(S-T)(z) = h(z)x_\alpha \qquad (z \in X)$$

where

$$h(z) = (1+\gamma^2)g(z) - T_1^*(f_\alpha)(z) + \eta w f_\alpha(z),$$

so $S-T$ is a rank-one operator and

$$\|S-T\| = \|h\| \leq \gamma^3 + \gamma^2 + \eta \leq 3\eta < \varepsilon.$$

∎

Now we use the result by Partington [12; Theorem 1] which asserts that every Banach space can be equivalently renormed to have property β and get

COROLLARY 5: *Every Banach space X is isomorphic to a Banach space Y in such a way that $\overline{R(Y)} = L(Y)$.*

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